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Correlating Pitch Levels and String Lengths in Iron-Strung Harpsichords*

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Wire stretched between two points and placed under tension becomes a vibrating string, and the factors governing the tension include its frequency, length, diameter, and density. The degree of tension is limited by the ultimate strength of the material; that is, for a given length, diameter, and density, there is a limit to how high the frequency may be raised before the string breaks. For a given tension, diameter and density, the frequency-length relationship is constant, so that to raise the frequency, the length must be shortened to prevent string breakage, and similarly, to lower the frequency, the length must be increased. A limiting factor at the upper range of the frequency-length relationship is that a string cannot be tensioned too close to its breaking point, since changes in climate conditions or manipulation of the string during tuning could break the string too easily. This requires that a safety factor be introduced, so that a string of a given length has a frequency a few semitones below its breaking point. Different wire materials have different strengths, so that the frequency-length relationship is different for different materials. Specifically, for a given diameter and length, a weaker material such as yellow brass cannot be raised to as high a frequency compared to a stronger material such as iron; similarly, for the same frequency, a yellow brass string must be shorter than an iron string. Different diameters of the same material also have slightly different strengths; thinner wire tends to be stronger as a consequence of how wire is manufactured (drawn through smaller and smaller holes to size it), a phenomenon known as tensile strength pickup.

Since a shorter string length has a proportionately higher pitch level than a longer string length, assuming all other factors are equal,[1] it

*I wish to thank Paul Irvin for reading an earlier draft of this essay and offering many useful comments and suggestions.

1. Grant O'Brien, *Ruckers: A Harpsichord and Virginal Building Tradition* (Cambridge: Cambridge University Press, 1990), 17, 56; and Denzil Wraight, "The Pitch Relationships of Venetian String Keyboard Instruments," in *Fiori Musicologici: Studi in Onore de Luigi Ferdinando Tagliavini*, ed. François Seydoux (Bologna: Pàtron Editore, 2001), 579–80.

should be possible to assign a pitch level to any harpsichord whose string lengths are known. The difficulty is trying to find a reference relationship between a certain length and a certain pitch with respect to historical harpsichords. In order to do this, accurate information is necessary about the other stringing factors involved. For example, one approach would be to establish the tensile strength of historical wire, determine the safety factors, and then calculate a pitch level from known string lengths.[2] The problem with respect to tensile strength is that there are a wide range of reported strength values of historical wire, and the decision as to which values to use for calculations would greatly affect the results.[3] The problem with respect to safety factors is that there is very little concrete evidence about what size historical makers chose or accepted.[4] Modern researchers either assume a certain safety factor[5] or use other information to calculate them.[6] However, it is possible to find evidence of pitch levels of historical iron-strung harpsichords from sources independent of tensile strength values and safety factors, and then correlate these pitch levels with the string lengths of those instruments.[7] From this information, an equation can be formulated that allows the calculation of an A-based pitch level from an instrument's c″ string lengths.

Preliminary Considerations

Pitch Level Notation. Note that "frequency" refers only to the number of cycles per second (such as 440 Hz) while "pitch" refers to a frequency

2. Cary Karp, *The Pitches of 18th Century Strung Keyboard Instruments, with Particular Reference to Swedish Material* (Stockholm: SMS-Musikmuseet, 1984), passim.

3. Thomas Donahue, "Evaluating Historical Stringing Information," *Early Keyboard Journal* 25/26 (2010): 125–51. My conclusion from evaluating thirty-seven reports of historical iron wire strength and thirty-five reports of historical brass wire strength was that no pattern emerges about the tensile strength or tensile strength pickup of old wire other than the possible ranges: 800–1200 MPa for iron wire and 700–1000 MPa for yellow brass wire.

4. Denzil Wraight, *The Stringing of Italian Keyboard Instruments c.1500–c.1800* (PhD diss., Belfast: Queen's University, 1997), 87–94.

5. Karp, *The Pitches*, 43: "The smallest allowable safety margin will be accepted here as one semitone."

6. R. Dean Anderson, "Michel Corrette and the Stringing, Scaling, and Pitch of French Harpsichords," *Early Keyboard Journal* 21 (2003): 73–76; and Michael Latcham, *The Stringing, Scaling and Pitch of* Hammerflügel *Built in the Southern German and Viennese Traditions 1780–1820* (Munich and Salzburg: Musikverlag Katzbichler, 2000), 86–87.

7. This approach has also been used by Darryl Martin, *The English Virginal* (PhD diss., Edinburgh: University of Edinburgh, 2003), 63–68.

associated with a specific note name (such as a′ = 440 Hz). A shorthand approach for notating pitch will be employed that has been used by Bruce Haynes.[8] The notation and the corresponding average pitch levels are shown in table 1.

fL. The specific relationship to be examined is the number obtained by multiplying the frequency of a note in hertz by the string length of that same note in meters. This will be referred to simply as "fL."[9] An fL value may be understood as a frequency-weighted string length, or length in the context of frequency. The units of measurements used here for fL are meters per second.[10] Representative fL curves for selected harpsichords are shown in figure 1; instruments are identified in table 2.

The next question is which note's fL value should be used. In terms of frequency, it makes no difference which note is selected, because frequencies cannot be "stretched" or "foreshortened" and therefore any

8. Bruce Haynes, *A History of Performing Pitch: The Story of "A"* (Lanham, MD: Scarecrow Press, 2002), lii. This classification system is different from the one he used in his dissertation. I have also used exact equal-tempered semitone values based on A440, which differ slightly from Haynes's values.

9. My previous discussions have referred to this as "fL-scale." This is in contrast to "fL-wire" which is an fL value derived from the tensile strength and density of wire; see Thomas Donahue, "A Method of Designing a Harpsichord String Plan," in *Music and Its Questions: Essays in Honor of Peter Williams*, ed. Thomas Donahue (Richmond, VA: OHS Press, 2007), 319–68; and Thomas Donahue, "Safety Factors for Replicas of the 1784 Hubert Fretted Clavichord in Edinburgh," in *De Clavicordio IX. Proceedings of the IX International Clavichord Symposium, Magnano 2009*, ed. Bernard Brauchli, Alberto Galazzo, and Judith Wardman (Magnano: Musica Antica a Magnano, 2010), 23–33. The concept of fL-scale may be found in E. O. Witt, "A Harpsichord Primer: One Maker's View," *Journal of the Audio Engineering Society* 23 (1975), 646–58.

10. Hertz is an alternate name for cycles per second, but the actual unit of measurement for frequency is "inverse seconds" or 1/sec. The reason for this is that a sound wave's frequency is the inverse of the wave's period. That is, period is the duration in seconds of one cycle of a repeating event, while frequency is the number of occurrences of a repeating event per second. For the low C on a harpsichord, the frequency is about 60 Hz and the period is 0.0167 seconds. The units of "meters per second" for fL may be recognizable as those for speed; there is a connection. The speed of a wave impulse in a harpsichord string is the frequency multiplied by the wavelength, and since the sounding length of a string is one-half the wavelength, there is a relationship between an fL value and the speed of the wave: an fL value is one-half the wave speed. A possible alternate unit for fL is "hertz-meters." My use of "meters per second" is based on the rationale of matching the units of measurements typically found in equations. In such equations, seconds are used, not hertz. Units of "hertz" and "newton" simplify discussions especially in prose, but working with measurements in basic units of meters, seconds, kilograms, and so on, promotes accuracy and allows one to better see relationships among physical factors.

TABLE 1. Pitch Level Notation, after Haynes.

A+1	A466
A+0	A440
A–1	A415
A–1½	A403
A–2	A392
A–3	A370

The numbers 1, 1½, 2, and so on, refer to the number of semitones above or below the reference pitch of A+0. Notice the convention of using "A440" in place of the more cumbersome "a′ = 440 Hz."

TABLE 2. Treble Scaling of Selected Harpsichords.

Note	Taskin	Hass	Kirckman	Ruckers
Deviation of treble fL values from c″ fL value, in cents				
f′	–22	–38	19	–27
c″	0	0	0	0
f″	–12	–6	–35	11
c‴	–24	13	–20	–19
c″-equivalent lengths				
f′	358.2	354.8	349.5	352.2
c″	363.0	363.0	346.0	358.0
f″	359.9	361.2	338.6	359.9
c‴	358.0	366.0	342.0	354.0

Divide cents by 100 to get semitones.
Instruments:
Pascal Taskin harpsichord, 1780, Musée de la Musique, Paris, no. E. 979.2.1
Johann Adolph Hass harpsichord, 1764, Edinburgh University Collection of Historic Musical Instruments, Edinburgh, no. 4314 or HS5.JH1764.14
Jacob Kirckman harpsichord, 1762, private collection
Andreas I Ruckers harpsichord, 1644a, Vleeshuis Museum, Antwerp, no. VH 2137

note's frequency can represent the pitch level. Even though there are some frequency variations due to different-sized intervals in various temperaments, these variations are small,[11] and this makes the use of frequencies based on equal-tempered semitones acceptable for the

11. Among the most common temperaments, the greatest deviation from equal-tempered semitones is in 1/4-comma meantone temperament, in which G-sharp deviates 27 cents from equal temperament when c′ = 261.6 Hz. In most other temperaments, the

calculations here. However, the decision as to which string length to use is not as simple, because string lengths do not have to follow a strict doubling-and-halving layout in order to sound acceptable. A commonly used reference string is c″, which will be useful for our current purpose, relying on an assumption about how historical makers designed their string plans. The assumption is that the makers would have known that for a given length, the frequency of a string can be raised only so far before breaking. Keeping a string of a given length below its breaking point by a few semitones established a "working length," and the lengths of other strings could be related to this first string by some relationship, typically that a string pitched an octave lower is twice as long.[12] This type of relationship—known as a just or Pythagorean progression—is found to a greater or lesser degree in the treble of many instruments and is manifested by the fact that the fL values are very similar in value, and when these values are graphed they form (with slight variation) a plateau in the curve (see fig. 1). The c″ string is often within this just (or almost-just) region, at least for harpsichords. If the reference string—the original "design" string—was also in this region, it would have an fL value fairly close in value to the c″ string, and so the c″ string would be representative of the design starting point. Table 2, for the same instruments in figure 1, shows how close the treble fL values are in relation to the c″ fL value, as well as how close the average of the c″-equivalent lengths for f′, f″, and c‴ are in relation to the actual c″ length. So for the present purpose, the c″ fL values will be used as a basis of comparison.

Sometimes stringing is discussed in terms of stress, which is defined as the (tensile) force per unit area to which the string is subjected. As it turns out, fL and stress are related by the expression:

$$\text{stress} = 4 \times \text{density} \times (\text{frequency} \times \text{length})^2$$

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deviations are generally no more than about 15 cents; see Thomas Donahue, *A Guide to Musical Temperament* (Lanham, MD: Scarecrow Press, 2005), 153–69.

12. Because of the phenomenon of tensile strength pickup in which thinner wire of the same material is stronger, there is a slight deviation from a just-progression. Because a string an octave lower than the reference string usually has a larger wire size of slightly lower strength, the string length may be slightly shorter than twice the reference length. Similarly, because a string an octave higher usually has a smaller wire size of slightly higher strength, that string length may be slightly longer than one-half the reference length. The degree to which historical makers modified a just-progression to accommodate tensile strength pickup is not clearly known because there are other things that may cause deviations from theoretical lengths, such as bridge shape and placement.

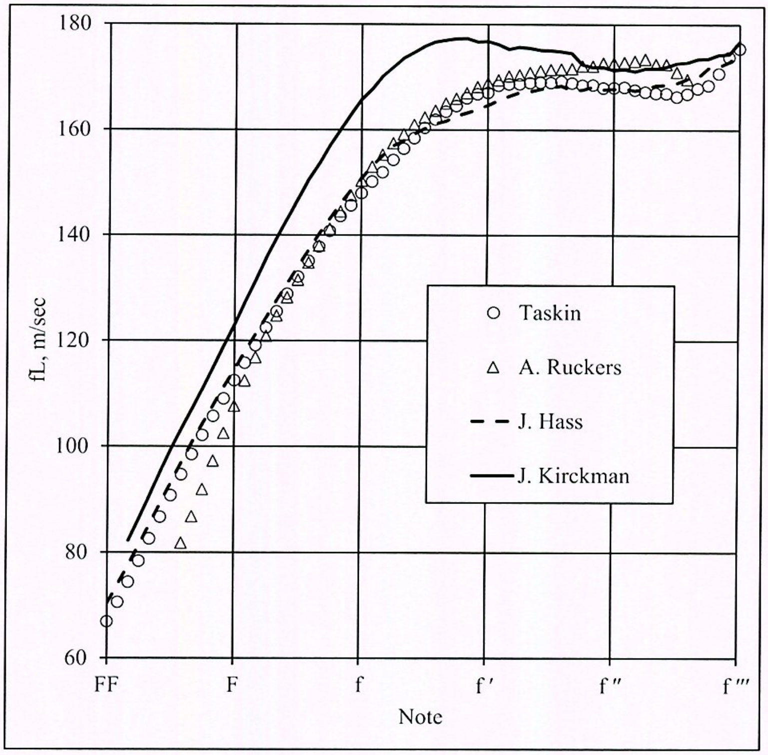

FIGURE 1. fL curves of selected harpsichords. The instruments are identified in table 1.

In other words, stress is proportional to fL squared. Either fL or stress could be used to study the subject. However, fL values are more useful for the present study for two reasons. First, frequency and length are the two quantities that are specifically being examined and correlated here. Second, there is the consideration of relating two quantities by using the relationship of the equal-tempered semitone, which is 100 cents and is based on the twelfth root of two or 1.0595 (see appendix). While semitones are usually used in reference to frequencies, they can also be used for relating string lengths; for example, a c″ length of 345 mm may be said to be 0.88 semitone shorter than a c″ length of 363 mm. (A good example illustrating the concept of "a length difference of one semitone" is the guitar: placing a finger on the first fret shortens the sounding length of the string by one semitone.) An advantage of using fL values is that the semitone relationship between them is the same magnitude as if one was dealing with two frequencies or two lengths. For example, with a c″ frequency of 466 Hz (A392), the two c″ lengths of 345 and 363 mm have fL values of 160.77 and 169.16 m/sec respectively. The semitone difference between these two fL values is 0.88, the same as the semitone difference of the lengths. In contrast, the stress on these two strings would be 806.4 and 892.8 MPa respectively. (MPa stands for megapascals, a unit of measurement for stress and tensile strength, in terms of force per unit area. A megapascal is one million newtons per square meter or one newton per square millimeter.) This is a semitone difference of 1.76—twice as much as the corresponding fL values—because stress is related to fL squared. This disparity in magnitude will be avoided here by using fL values rather than string stress.

Associating Pitches and Lengths

There is evidence of known pitch levels that can be linked with specific harpsichords whose string lengths are known. This documentation will allow an examination of the frequency-length relationships.[13] Around 1682 in Amsterdam, Christiaan Huygens did an experiment in which he produced a tone that corresponded to d″ on his harpsichord. The frequency of the tone was calculated to be 547 cycles per second, which corresponds to a pitch level of A409/C486, assuming D-A is a tempered fifth.[14] The scaling practice of Ioannes Couchet is instructive for a potentially associated string length, since Huygens's father is known to have owned a harpsichord by that maker.[15] The c″ lengths in ten Couchet instruments have a range of 327 to 368 mm,[16] which gives a

13. With respect to the calculations done here, string lengths in millimeters and frequencies in hertz are rounded to the nearest whole number, while fL values are rounded to one decimal place. Note that c″ fL values are calculated by using the c″ frequency, not the a′ frequency; the c″ frequency is the a′ frequency multiplied by 1.189. Also note that the fL values use the c″ string length in meters.

14. Sigalia Dostrovsky, "Early Vibration Theory: Physics and Music in the Seventeenth Century," *Archive for History of Exact Sciences* 14 (1975): 197–204; Bruce Haynes, *Pitch Standards in the Baroque and Classical Periods* (PhD dissertation, University of Montreal, 1995), 550–58; and Karp, "Pitch," 159–62.

15. O'Brien, *Ruckers*, 305–8.

16. Donald H. Boalch, *Makers of the Harpsichord and Clavichord 1440–1840*, 3rd ed., edited by Charles Mould (Oxford: Clarendon Press, 1995), 273–78; and O'Brien, *Ruckers*, 271–76.

range of fL values of 158.9–178.8 m/sec. Unfortunately, it is not clear whether each of these values represent the longer c″ length. If Couchet's practice is seen as closely following Ruckers's practice and we take the average Ruckers scaling of 355 mm,[17] this gives an average c″ fL value of 172.5 m/sec.

A two-manual harpsichord by Johann Heinrich Gräbner the younger (ca. 1700–1777), made in 1739 and now in the Kunstgewerbemuseum Dresden at Schloss Pillnitz (no. 37414), is a five-octave, 8.8.4 instrument.[18] It has the unusual range of DD–d‴, which apparently is the original unaltered compass. A more common range for that time and place would have been FF–d‴. There could be many reasons for the unusual range, such as, it presents a symmetrical layout of the sharps. However, an intriguing explanation suggested by Christian Ahrens[19] is that this harpsichord was a transposing instrument, although not in the sense of having a sideways-shifting keyboard. In other words, the DD-d‴ instrument was meant to be used in unison with a C-based instrument pitched one whole tone higher. This suggests its use as a continuo instrument in conjunction with an organ in which the harpsichord's DD note pitched at Cammerton doubled the 16′ C in the pedal division of an organ pitched at Chorton. Chorton in Dresden at that time would have been A+1 (averaging about A466),[20] meaning the harpsichord at Cammerton would have been pitched at A415/C493. (While Dresden was noted for having many of its organs tuned to Cammerton, there were organs at both pitches.[21]) The c″ length of the 1739 instrument is 339 mm and is similar to other instruments of the Gräbner family: 336.5 mm in the 1774 harpsichord by the same maker and 336 mm in the 1782 instrument by his son Carl August.[22] Given this data, we can calculate a c″ fL of the 1739 instrument as 167.1 m/sec.

While Cammerton typically was related to harpsichords and not organs, in Hamburg there were two documented instances of Cammerton

17. O'Brien, *Ruckers*, 61.

18. John Phillips, "The 1739 Johann Heinrich Gräbner Harpsichord: An Oddity or a *Bach-Flügel*?" in *Das Deutsche Cembalo*, ed. Christian Ahrens and Gregor Klinke (Munich: Katzbichler, 2000), 123–39.

19. Phillips, "The 1739 Johann Heinrich Gräbner Harpsichord," 131.

20. Haynes, *Pitch Standards*, 233–36.

21. Haynes, *Pitch Standards*, 234–35.

22. The 1722 harpsichord by Johann Heinrich Gräbner the elder has a c″ length of 348 mm. This is a reconstructed length because the nut had been moved, but it is still less than one-half semitone longer than the 1739 c″ length.

in relation to organs. This is significant because the pitch of organs is generally a reliable documentation of a pitch level. The first example is the 8′ Gedackt stop in the 1693 Schnitger organ in St. Jacobi Church, Hamburg, which was at Cammerton even though the rest of the organ was at Chorton. This stop was mentioned in the church archives and was also cited by Mattheson in 1721 and by Adlung in 1758 and 1768.[23] The organ was pitched at A489 and the Gedackt stop was pitched a minor third lower, at A411/C489.[24] The second example is the 1767 organ in the Michaeliskirche, Hamburg, by J. G. Hildebrandt which was at Cammerton, reported to be A408/C485.[25] However, there are examples of different Hamburg harpsichord scalings about one semitone apart, so the question arises as to which lengths would be at Cammerton. For example, the longer c″ lengths of two Christian Zell harpsichords are in the range of 344 to 349 mm, while the longer c″ lengths of several Hass- and Fleischer-family instruments are on the order of 359 to 367 mm. The average difference between these two groups (363 vs. 346) is 83 cents (table 3).

One possible explanation of this scaling difference is the existence of more than one Cammerton,[26] or, perhaps more precisely, the term "Cammerton" encompassed more than one pitch level. For example, *tief-Cammerton* ("low chamber pitch") has been documented and may represent a pitch as low as A390/C464.[27] Given this possibility, it is probably more logical to coordinate the longer Hass/Fleischer scalings with this lower pitch level and the Zell scalings with A408–411. In this way, both scalings are within the context of Cammerton; that is, no higher than A–1. With the c″ length of 349 mm from the 1728 Zell instrument, assuming a pitch of A410/C487 gives a value of 168.4 m/sec. With the c″ length of 363 mm from the 1764 J. Hass instrument, assuming a pitch range of A390/C464 gives a value of 168.4 m/sec.

A valuable linking of a c″ length with a pitch level may be found with several claviorgana. One is the Earl of Wemyss's instrument dating from about 1745–51, which pairs a Jacob Kirckman harpsichord with a John

23. Haynes, *Pitch Standards*, 240–41.

24. Alexander J. Ellis, "The History of Musical Pitch," (originally *Journal of the Society of Arts*, 5 March 1880; reprint London: Trounce, 1880), 318, 331, 335; and Haynes, *Pitch Standards*, 240.

25. Ellis, "History," 318, 335.

26. Haynes, *Pitch Standards*, 209–32.

27. Haynes, *Pitch Standards*, 227.

TABLE 3. Selected Harpsichords by Eighteenth-Century Makers in Hamburg.

Maker	Year	Disposition	c″ Length, mm
C. C. Fleischer	1720	I 8.8.4	(367)
J. C. Fleischer	1710	I 8.8.4	(361)
H. Hass	1723	II 8.8.8.4	366.5
H. Hass	1726(A)	I 8.4	364
H. Hass	1732	I 8.8.4	(361)
H. Hass	1734	II 16.8.8.4	359
H. Hass	1740	III 16.8.8.4.2	359
J. Hass	1760	II 16.8.8.4.2	?
J. Hass	1764	I 8.8.4	363
C. Zell	1728	II 8.8.4	346, 349
C. Zell	1737	I 8.8.4	344

A c″ string length in parenthesis means it has not been verified as the longer 8′.
Fleischer 1720: Museu de la Musica, Barcelona, no. MDMB 420
Fleischer 1710: Musikinstrumenten Museum, Berlin, no. 5083
Hass 1723: Musikmuseet, Copenhagen, no. A48 (length verified by Ture Bergstrøm, personal communication, 12 October 2009)
Hass 1726(A): Leufsta Bruk Manor, Leufsta
Hass 1732: Nasjonalmuseet, Oslo, no. 10780
Hass 1734: Brussels Museum, Brussels, no. 630 (length verified by Andrea Goble, personal communication, 24 September 2009)
Hass 1740: private collection, France (length verified by Andrea Goble, personal communication, 24 September 2009)
Hass 1760: Yale University, New Haven, no. 4879.60
Hass 1764: Edinburgh University Collection of Musical Instruments, Edinburgh, no. 4314 or HS5-JH1764.14 (length verified on collection's Web site)
Zell 1728: Museum für Kunste und Gewerbe, Hamburg, no. 1962.115 (The 346-mm length was verified by Andrea Goble, personal communication, 24 September 2009. The 349-mm length was the measurement taken in 1973 when the first restoration was done, verified by Martin Skowroneck, personal communication, 16 September 2009)
Zell 1737: Museu de la Musica, Barcelona, no. MDMB 418 (length verified by Andrea Goble, personal communication, 24 September 2009)

Snetzler organ.[28] The longer c″ string length is 337 mm. The actual pitch of the organ has not been verified, but several pieces of evidence offer clues. First, the pitch levels of several surviving Snetzler organs are

28. Peter Williams, "The Earl of Wemyss' Claviorgan and its Context in Eighteenth-Century England," in *Keyboard Instruments: Studies in Keyboard Organology, 1500–1800*, ed. Edwin M. Ripin (Reprint, New York: Dover, 1977), 77–87.

known (table 4). Second, Haynes found a consistent pitch level in English church organs from 1660 onwards, ranging from 420 to 430 Hz (averaging 425.5 Hz) and secular organs tending to be pitched similarly after 1740.[29] Using a pitch level of A425/C505 and a string length of 337 mm, an approximate c″ fL value for this instrument would be 170.2 m/sec. Another claviorganum is that of Lodewijk Theewes from 1579. From a surviving wooden pipe, John Koster estimates the organ's pitch at either one semitone below A440 (that is, A415) if the pipe was originally stopped, or 1½ to 2 semitones below A440 (403 to 392 Hz) if the pipe was originally open.[30] The c″ string length has been given variously as 346 mm,[31] 349 mm,[32] 350 mm,[33] and 356 mm.[34] A detailed analysis accounting for the differences in c″ length has yet to be presented; the 350-mm value is the most-quoted and is also the average of the four values. Using a pitch range of A392/C466 to A415/C493 and a c″ length range of 346 to 356 mm, a range of c″ fL values is 161.2–175.5 m/sec. Using average values of A403/C479 and 350 mm gives an average c″ fL value of 167.7 m/sec.

Yet another claviorganum is that of Lorenz Hauslaib from 1598, a combination of a small four-stop organ and a pentagonal *octave* spinet.[35] The c″ length is 160 mm, strung in iron, and the remaining pipes suggest an equivalent pitch level of A443–448. Using this range, the frequency of the spinet's c″ would be 1053–1065 Hz (being pitched an octave higher) which yields a c″ fL range of 168.5–170.4 m/sec or an average of 169.4

29. Haynes, *Pitch Standards*, 329.

30. John Koster, "The Importance of the Early English Harpsichord," *Galpin Society Journal* 33 (1980): 53–54.

31. Malcolm Rose, "Further on the Lodewijk Theewes Harpsichord," *Galpin Society Journal* 55 (2002): 290.

32. Martin, *The English Virginal*, 332.

33. Wilson Barry, "The Lodewijk Theewes Claviorganum and its Position in the History of Keyboard Instruments," *Journal of the American Musical Instrument Society* 16 (1990): 17; Boalch/Mould, *Makers*, 355; Koster, "The Importance," 51; and Howard Schott, *Victoria and Albert Museum Catalogue of Musical Instruments*, Vol.1, *Keyboard Instruments*, 2nd ed. (London: Her Majesty's Stationery Office, 1985), 40.

34. Edwin M. Ripin et al., *The New Grove Early Keyboard Instruments* (New York: W. W. Norton, 1989), 28.

35. Stewart Pollens, "The Claviorgan by Lorenz Hauslaib, Nuremberg, 1598," in *Das Österreichische Cembalo: 600 Jahre Cembalobau in Österreich* (Tutzing: Hans Schneider, 2001), 247–67.

TABLE 4. Pitch Levels of Organs by John Snetzler.

Year	Location	Pitch
1740	Chapel Royal, St. James Place, London	A426[a]
1759	Cobbe Collection, Hatchlands Surrey	A425[b]
1761	Smithsonian Institution, Washington, D.C.	A420±[c] or ~A427[d]
1764	Halifax Parish Church, West Yorkshire	A426[e]
1766	Kedleston Hall, Derbyshire	A425[f]

a. Alexander J. Ellis, "The History of Musical Pitch," (originally *Journal of the Society of Arts*, 5 March 1880; reprint, London: Trounce, 1880): 321.

b. Martin Goetze and Dominic Gwynn, "Restoration of the 1759 Snetzler Chamber Organ," http://www.goetzegwynn.co.uk/restored/hatchlands.shtml (accessed 27 September 2009).

c. Bruce Haynes, *Pitch Standards in the Baroque and Classical Periods* (PhD diss., Montreal: University of Montreal, 1995), 536.

d. John T. Fesperman, *A Snetzler Chamber Organ of 1761* (Washington, D.C.: Smithsonian Institution Press, 1970), 38.

e. Ellis, "The History of Musical Pitch," 321.

f. Christopher Kent, "Temperament and Pitch," in *The Cambridge Companion to the Organ*, ed. Nicholas Thistlewaite (Cambridge: Cambridge University Press, 1998), 52.

m/sec. Data from several other claviorgana are given by Darryl Martin from information supplied by Alfons Huber[36] (table 5).

In his examination of harpsichords of the Blanchet family, William Dowd suggests that certain features of the 1730 and 1733 instruments point to the possibility that each may have been intended to be part of a claviorganum.[37] This conclusion is based on certain features such as shallow cases, a wide octave spacing—perhaps to accommodate a rollerboard—and slides projecting through the cheeks, possibly to be connected to a stop control. The shorter c″ string lengths are 349 mm (1730) and 342 mm (1733); the longer c″ lengths are probably on the order of 367 and 360 mm. Documented pitches of sixteen French organs from the period 1710 to 1730 range from 385 to 415 Hz, with an average of 400 Hz.[38] The range of c″ fL values would be 164.9–180.9 m/sec. With a pitch level of A400/C476 and a c″ length of 364 mm, this gives an average fL of 173.3 m/sec.

36. Martin, *The English Virginal*, 64–65.

37. William Dowd, "The Surviving Instruments of the Blanchet Workshop," in *The Historical Harpsichord*, vol. I, ed. Howard Schott (Stuyvesant, NY: Pendragon Press, 1984), 60.

38. Haynes, *Pitch Standards*, 508–10.

TABLE 5. Scale and Pitch Information on Selected Claviorgana.

Maker	Year	c″ Length, mm	Pitch, Hz	c″ fL m/sec
Rorif	1565–69	324	ca. A415	159.7
Pock	1591	306	A465	169.2
Zeiss	1639	315	A460	172.3
Zeiss	1646	300	A460	164.1

Data from Darryl Martin, *The English Virginal* (PhD diss., Edinburgh: University of Edinburgh, 2003): 64–65.
Servatius Rorif, 1564–59, Kunsthistorisches Museum, Vienna
Josua Pock, 1591, Dommuseum, Salzburg
Valentin Zeiss, 1639, Museum Carolino-Augusteum, Salzburg
Valentin Zeiss, 1646, private collection, Austria

An interesting linking of string length with pitch involves the existence of a tuning fork found with a 1791 clavichord by Pehr Lindholm.[39] This instrument has many old strings and points to iron stringing in the treble. Its c″ length is 316 mm. The association of this length with iron stringing is corroborated by the research of Eva Helenius-Öberg (the average length among twenty-two Lindholm clavichords from the 1780s and 1790s is 322 mm, and documented stringing schemes demonstrate iron in the treble[40]) and Bernard Brauchli ("the scaling [of eighteenth-century Swedish clavichords] . . . was generally long, with c″ = 31 or 32 cm, intended for iron strings except in the bass"[41]). The association of the instrument with the tuning fork is suggestive, but it is possible it could have been placed with the instrument at a later time. The tuning fork has a frequency of 463.3 Hz, and assuming it was used for a′, c″ would be 551 Hz, giving a c″ fL value of 174.1 m/sec.

Pehr Lundborg's "organized clavichord" from 1772 (serial no. 37; Stockholm Musikmuseet no. 1440) is a clavichord with a 4′ organ stop

39. Grant O'Brien, "The Lindholm Clavichord, Stockholm 1791," in *De Clavicordio VII: The Clavichord and the Lute. Proceedings of the VII International Clavichord Symposium, Magnano 2005*, ed. Bernard Brauchli, Alberto Galazzo, and Judith Wardman (Magnano: Musica Antica a Magnano, 2006), 42–43.

40. Eva Helenius-Öberg, *Svenskt Klavikordbygge 1720–1820* (Stockholm: Almqvist & Wiksell, 1986), 279, 283.

41. Bernard Brauchli, *The Clavichord* (Cambridge: Cambridge University Press, 1998), 18.

playable from the same keyboard. The organ pitch is about A460/C547[42] and the c″ length is 276 mm, strung in iron.[43] This gives a c″ fL value of 151.0 m/sec.

A pitch-length relationship may also be detailed by examining the large number of surviving harpsichords of similar design from an area with a well-documented pitch level. One such possibility is eighteenth-century England. In terms of string lengths, scaling information[44] with respect to twenty-four instruments by Jacob Kirckman, fifteen instruments by Abraham and Jacob Kirckman, and sixteen instruments by Shudi and Broadwood demonstrate an average c″ length of 344 mm, with 70% of the lengths ranging from 340 to 349 mm. In terms of the pitch level at that time, several pieces of evidence are available: (1) Handel's tuning fork from 1751 for a *Messiah* performance has a frequency of 422.5 Hz;[45] (2) a Stadden pitchpipe from 1774 has a frequency of 425 Hz;[46] (3) two tuning forks from the Broadwood workshop from ca. 1800 have frequencies of 422.7 and 423.6 Hz;[47] and (4) a written comment by Handel in a musical manuscript mentioned that strings and voices played at a different pitch level, which was one whole tone lower than an organ known to have been pitched at about A474 (making the lower pitch A422).[48] With frequencies of A422 to A425 and c″ lengths of 340 to 349 mm, the range of c″ fL values would be 170.7 to 176.2 m/sec. With averages of A423 and 344 mm, this yields a c″ fL value of 173.0 m/sec. Darryl Martin's research led him to similar pitch and length values.[49]

Michael Praetorius wrote in 1618 that the sixteenth-century Flemish builder Hans Bos pitched his instruments a minor third lower than the Cammerton of the time.[50] Praetorius's Cammerton was A+1 which places

42. Jenny Nex and Lance Whitehead, "A Preliminary Investigation into the Stringing of Swedish Clavichords," in *De Clavicordio IV: Proceedings of the IV International Clavichord Symposium, Magnano 1999*, ed. Bernard Brauchli, Susan Brauchli, and Alberto Galazzo (Magnano: Musica Antica a Magnano, 2000), 150.

43. The wire material for this note was verified to the author by Dan Johansson, curator of the Stockholm Musikmuseet (personal communication, 29 September 2009).

44. Boalch/Mould, *Makers*, 422–457, 613–26.

45. Ellis, "History," 319–20.

46. Haynes, *Pitch Standards*, 331.

47. Ellis, "History," 320.

48. Haynes, *Pitch Standards*, 330–31.

49. Martin, *The English Virginal*, 64–65.

50. See Haynes, *Pitch Standards*, 392n29; Karp, "Pitch," 153; and Martin, *The English Virginal*, 67n53.

the Flemish pitch at A–2. Only one Bos instrument survives, a 1578 virginal with a c″ length of 386 mm.[51] Using an average pitch for A–2 of A392/C466 gives a c″ fL value of 179.9 m/sec.[52]

The c″ fL values calculated above are summarized in table 6. Also included are stress values for comparison, which may be calculated from the fL data (see appendix). Both the range of values and a selected average for each source are listed. Note that the total averages of the first two columns are essentially the same (a difference of nine cents). The values in brackets were excluded from the total average because they are inconsistent with the other data and will be discussed later. There are, of course, certain caveats when dealing with such calculations:

1. Averaging and estimation of numbers would influence the final values.
2. A number value may be incorrect due to confusion between the long and short c″ lengths, faulty arithmetic, incorrect transcription of numbers, or typographical errors.
3. It is possible that an assumption may be wrong; for example, with respect to a tuning fork: that it represents the note A, that it was used to tune the note A on an instrument, and that it is correctly associated with a certain instrument or a particular circumstance.
4. There could be an incorrect association of a pitch level with a specific instrument.
5. It should be remembered that pitch levels could represent a range of frequencies as opposed to one single definitive frequency.

Given such possibilities, all data should be considered provisional pending further study. Nevertheless, the consistency of thirteen of the sixteen sources is noteworthy.

Estimating Pitch Levels from String Lengths

Based on the evidence presented, an equation can be devised that may be used to assign an approximate pitch level to a harpsichord with known string lengths. Since discussions about stringing often involve the

51. John Koster, "Three Early Transposing Two-Manual Harpsichords of the Antwerp School," *Galpin Society Journal* 57 (2004): 93.

52. The association of this pitch and this length has already been documented by John Koster, as reported by Darryl Martin, *The English Virginal*, 64.

TABLE 6. c″ fL and Stress Values for Iron Wire, Ordered by Frequency.

Source	Range of c″ fL, m/sec	Average c″ fL, m/sec	Range of Stress, MPa	Average Stress, MPa
Hass	168.4	168.4	885	885
Bos	179.9	[179.9]	1009	[1009]
Blanchet	164.9–180.9	173.3	848–1021	937
Theewes	161.2–175.5	167.7	811–961	877
Huygens	158.9–178.8	172.5	788–998	929
Zell	169.3–170.7	170.0	894–909	901
Gräbner	167.1	167.1	871	871
Rorif	159.7	[159.7]	796	[796]
Kirckman et al.	170.7–176.2	173.0	909–969	934
Kirckman/Snetzler	170.2	170.2	904	904
Hauslaib	168.5–170.4	169.4	886–906	896
Lundborg	151.0	[151.0]	711	[711]
Zeiss 1639	172.3	172.3	926	926
Zeiss 1646	164.1	164.1	840	840
Lindholm	174.1	174.1	946	946
Pock	169.2	169.2	893	893
Total Average	169.2 ± 1.56 ST	170.1 ± 0.51 ST	895	903

ST = semitone(s)

The total average for the range of values was calculated using the high and low values of the range.

The numbers in brackets are excluded from the total average.

frequency of a′ and the length of c″, these will be the two elements used in the equation. Because the frequency of c″ was used to calculate the fL values above, the 170.1 value needs to be converted to an a′-based value by using the –3 semitone multiplier, which is 0.8409. This yields 143.037. Therefore:

EQUATION 1 (a′ frequency in Hz) × (c″ length in *meters*) = 143.0 m/sec

EQUATION 2 (a′ frequency in Hz) × (c″ length in *millimeters*) = 143037 mm/sec

With these equations, a table may be constructed that correlates A-based pitch levels with the longer c″ string lengths of iron-strung harpsichords on an average basis (table 7). Some example instruments are included in

TABLE 7. Correlation of Pitch Levels and c″ String Lengths for Iron-Strung Harpsichords.

Classification	Average a′ Freq.	Average c″ Length	Example (Maker, Longer c″ Length)
A+½	453 Hz	316 mm	Coston ca. 1725 harpsichord, 314 mm
A+0	440	325	van Everbroeck 1659 harpsichord, 323 mm
A–½	427	335	Anonymous 1667 harpsichord, 334 mm
A–1	415	345	Taskin 1788 harpsichord, 345 mm
A–1½	403	355	Ruckers average, 355 mm
A–2	392	365	Taskin 1780 harpsichord, 363 mm
A–2½	380	376	Hemsch ca. 1736 harpsichord, 378 mm
A–3	370	387	Richard 1688 harpsichord, 388 mm

Francis Coston harpsichord, ca. 1725, Edinburgh University Collection of Historic Musical Instruments, no. 4320 or HD3-FC1725.20
Gommaar van Everbroeck harpsichord, 1659, National Music Museum, no. 3985
Anonymous harpsichord, Paris, 1667, Museum of Fine Arts, Boston, no. 1977.55
Pascal Taskin harpsichord (attributed), 1788, Museo del Castello Sforzesco, Milan, no. 604
Ruckers average from Grant O'Brien, *Ruckers: A Harpsichord and Virginal Building Tradition* (Cambridge: Cambridge University Press, 1990), 61.
Pascal Taskin harpsichord, 1780, Musée de la Musique, Paris, no. E. 979.2.1
Henri Hemsch harpsichord, ca. 1736, Museum of Fine Arts, Boston, no. 1981.747
Michel Richard harpsichord, 1688, Rhode Island School of Design, on loan to Yale University

the table to demonstrate this. Accepting the average c″ fL values from table 6 and excluding the inconsistent data, then the range of ± ½ semitone would also apply to the information in table 7; for example, a pitch of A415 may refer to string lengths of 345 ± 10 mm, and a c″ length of 345 mm may refer to frequencies of 415 ± 12 Hz.

The separation of the pitch levels in table 7 into half-semitone increments does not necessarily mean that instruments were scaled to this tolerance. Organizing the list by half-semitones is a consequence of the available data on surviving instruments, and is probably a result of the lack of precise historical pitch standardization by frequency as known today and that the factors associated with vibrating strings have a range of values. The string lengths of some surviving instruments suggest scaling by semitone or even half-semitone (see the Taskin instruments discussed below), but often the available scaling information involves a sample size that is too small, or a collection of data in which no pattern of pitch groups emerges. Looking at the data collected by Anderson on fifty-six seventeenth- and eighteenth-century French harpsichords (fig. 2), there appears to be no definite groupings between c″ lengths of 345 to 378 mm, a range of more than 1½ semitones.[53] This means that, with information on only one instrument with a c″ length of, say, 355 mm, one can never be sure whether such scaling is at the center of its own scaling group, at the high end of a shorter-scaled group, or at the low end of a longer-scaled group. Nevertheless, the existence of half-semitone pitch levels illustrated by the existence of A–1½ in eighteenth-century France and A–½ in eighteenth-century England[54] makes the pitch classification in table 7 a useful system. However, it is necessary to accept the fact that if there is any ambiguity in assigning a given c″ length to a particular "length group," variation in the pitch level of at least ± ½ semitone must be allowed.

Other Studies

Studies by other researchers corroborate these findings. Denzil Wraight has studied early Italian stringed keyboard instruments and has specifically investigated the pitch-length relationship of Venetian instruments, since those instruments have survived in the greatest number.[55] He coordinated pitch levels with the string lengths of sixty-two instruments: twenty-six harpsichords, thirty-five virginals, and one clavichord. The basis for the association is his examination of the tensile strength of modern and historical wire, in which he concludes that a c″ string with a length of about 339 mm in iron wire (or about 283 mm in brass wire) could have stood at about A413/C491.[56] Since this figure includes a safety factor, it is directly comparable with the values given above and gives a c″ fL value for iron wire of 166.4 m/sec. The calculated average of 170.1 m/sec is only 38 cents higher than this.

Another study is the examination of English virginals by Darryl Martin.[57] His approach was the same as that employed here: to associate string lengths of a specific instrument or group of instruments with independent pitch determinations. He then applied this information to the

53. Anderson. "Extant Harpsichords, Part I," table 6, unmarked pages 109–11.

54. Haynes, *A History of Performing Pitch*, 370, 378. Haynes uses the alternate designation of Q–2 for a pitch level in England, which is roughly equivalent to A–½.

55. Wraight, "The Pitch Relationships," 573–604.

56. He actually mentions a c″ frequency of 498 Hz, which is A418. We use A413/C491 here because that is the value used in his calculations.

57. Martin, *The English Virginal*.

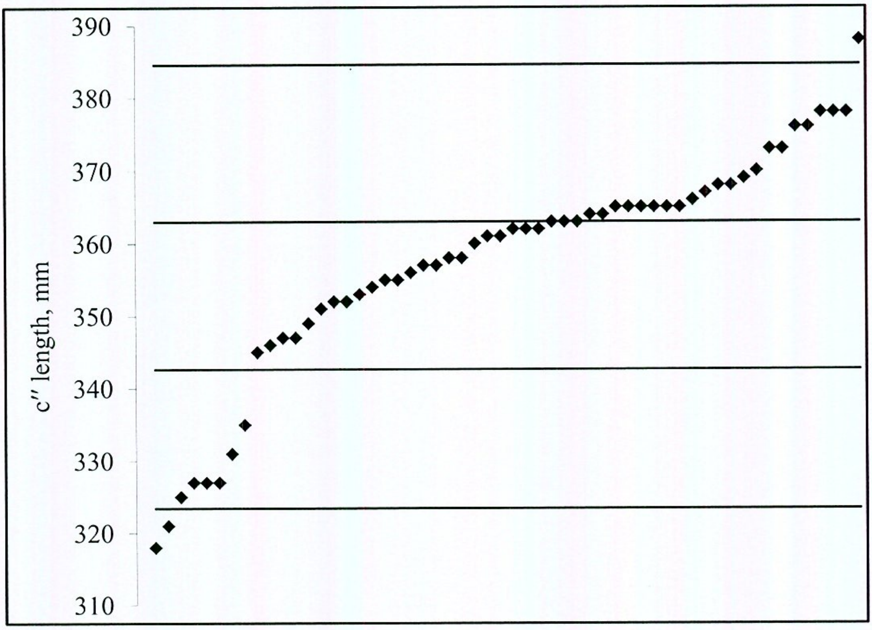

FIGURE 2. The c″ string lengths of fifty-six seventeenth- and eighteenth-century French harpsichords. The horizontal lines designate increments of one semitone. Derived from the data in R. Dean Anderson, "Extant Harpsichords Built or Rebuilt in France during the Seventeenth and Eighteenth Centuries: An Overview and Annotated List, Part 1," *Early Keyboard Journal* 19 (2001): table 6, unmarked pages 109–11.

string lengths of twenty-two surviving English virginals. He found a range of pitch levels of A467 to A485 (average A474/C564) for a twelve-inch scale. Of the twenty-two instruments, according to Martin's research, the design string for one instrument was a twelve-inch c″, for seven instruments was a six-inch c‴, and for three instruments was a four-inch f‴. Converting all these to c″-equivalent lengths and averaging them gives a c″ average length of 303 mm. With an average pitch level of A474/C564, this gives a c″ fL value of 170.9 m/sec. The calculated average of 170.1 m/sec is only eight cents lower than this.

Alfons Huber proposed a guideline relating string length and tensile strength: he takes the c″-equivalent length of a given note in *centimeters* and squares it. The result is the approximate tensile strength in MPa necessary to reach a pitch of A415 with a two-semitone safety factor (or A440 with a one-semitone safety factor).[58] According to equation 2 above, a pitch of A415 is associated with an average c″ length of 345 mm (or 34.5 cm). Squaring this gives a tensile strength value of 1190 MPa. This is equivalent to an fL-*wire* value (in contradistinction to the fL-*scale* values used above) of 195.3 m/sec as given by the equation

$$\text{fL - wire} = \sqrt{\frac{\text{T.S.}}{4\rho}}$$

in which "T.S." is the tensile strength in N/m^2 and ρ is the density of the wire material in kg/m^3. However, the fL-wire value is a maximum related to the ultimate strength of the wire, not the lower "working" value with a safety factor. Accounting for a two-semitone safety factor by multiplying by 0.8909 (the –2 semitone multiplier) gives a value of 174.0 m/sec. The calculated average of 170.1 m/sec is only 39 cents lower than this.

Consistent vs. Inconsistent Data

The use of the term "consistent" in the present context refers to the thirteen average fL values that are within ± ½ semitone of each other, while the "inconsistent" data are the three values that are outside this range. In general, an fL value lower than the average is the result of lower frequencies or shorter lengths, while an fL value larger than average is the result of higher frequencies or longer lengths. Some research-related reasons already been mentioned may contribute to the inconsistency; perhaps the most probable reason is that the pitch level assigned to any given instrument is incorrect. However, there are other design-related aspects that would account for these variations in fL values.

The first is wire strength. Stronger wire allows a string of a given length to stand at a higher pitch, or a string of a given pitch to be longer, assuming all other factors are equal. In the first case, the frequency is higher; in the second case, the length is longer; each of which results in a higher fL value. Weaker wire requires a string of a given length to stand at a lower pitch, or a string of a given pitch to be shorter, assuming all other factors are equal. In the first case, the frequency is lower; in the second case, the length is shorter; each of which result in a lower fL value. Note that stronger wire *in and of itself* does not increase the fL value. Rather, a higher fL value follows from stronger wire if the maker

58. Alfons Huber, "Iron Scale or Brass Scale: When Were These Concepts First Used?" in *De Clavicordio VI: Proceedings of the VI International Clavichord Symposium, Magnano, 2003* (Magnano: Musica Antica a Magnano, 2004), 27.

has understood the properties of the wire and used its additional strength in the form of higher pitch or longer length. Stronger wire could refer to either overall stronger wire (that is, the batches of wire from manufacturer A have greater strength than the batches from manufacturer B) or to tensile strength pickup, the phenomenon that thinner wire of the same material is stronger.

Another aspect is the safety factor. A relatively larger safety factor—a string is set farther from its breaking point—means a string of a given length is tuned lower or a string at a given pitch is shorter compared to a string with a smaller safety factor. Each of these situations results in a lower fL value. A relatively smaller safety factor—a string is set closer to its breaking point—means a string of a given length is tuned higher or a string at a given pitch is longer compared to a string with a larger safety factor. Each of these situations results in a higher fL value.

The effect of wire strength and safety factors can either reinforce or counteract each other. For example, stronger wire (if accounted for) and smaller safety factors can raise fL values, while stronger wire and larger safety factors would offset each other. Unfortunately, having an average fL value and multiple variables does not provide any information about what weight each variable may have been given in the design process. It is interesting to speculate that the consistency of thirteen of the sixteen average fL values given above (± ½ semitone) is an indicator of the use of similar strength wire and a small range of safety factors. This seems logical, if only because alternative possibilities do not seem likely, such as: (1) that similar c″ fL values are a result of makers using greater safety factors with higher-strength wire and smaller safety factors with lower-strength wire; or (2) that values for pitch, length, tensile strength, and safety factors were uncoordinated or random, and the resultant similar c″ fL values are a statistical coincidence.

A good example of the interplay of wire strength and safety factors may be seen with the 1772 Lundborg organized clavichord with a c″ fL value of 151.0 m/sec, two semitones below the average. Since the instrument is associated with an organ stop with a well-defined pitch, the pitch is apparently incontestable, so the explanation seemingly lies with matters of stringing. One aspect is the fact that clavichords tend to be strung heavier than harpsichords; for example, gauge 5 is found for the note c″ on seven surviving Lundborg clavichords.[59] Larger wire sizes have

59. Helenius-Öberg, *Svenskt Klavikordbygge*, 284.

slightly lower tensile strengths than their thinner counterparts because of tensile pickup, and as a consequence, weaker wire needs to have shorter lengths. Another aspect is that clavichord strings are "attacked" by the tangent and are stretched more in playing compared with harpsichords. This would make larger safety factors more appropriate. For a given pitch, one way to get larger safety factors is to use a shorter string length.

Another design aspect that would produce inconsistent data has to do with non-Pythagorean layouts; for example, C or F lengths that are not in a strict halving-and-doubling relationship. If a maker did not use c″ as a design starting-point and if the relationships of other string lengths to the reference length deviated significantly from Pythagorean—that is, there is not a well-defined plateau in the fL curve—this would mean that the c″ length (and therefore the c″ fL value) may not be a good indicator of the treble pitch-length relationship for that instrument.

Taskin's Tuning Fork

Even though there are several surviving instruments by Pascal Taskin and a tuning fork pitched at A409 that is associated with him, this information was intentionally excluded in the discussion above.The following entry is taken from Ellis:[60]

> A409 . . . 1783, Paris, Court Clavecins. Fork of Pascal Taskin, their tuner, tuned from A of oboe of Antoine Sallentin, of the Opera and Chapelle du Roi, given by Taskin to M. Pfeiffer, who possessed it in 1859, according to de la Fage.

The problem is, the surviving instruments of Taskin show diverse scalings. The longer c″ string lengths of thirteen surviving instruments arranged arbitrarily are:

303	344	357	360	366
	345	357	362	369
	345		363	371
			363	

Taskin instruments were not included in the discussion above because there is no independent way of verifying to which c″ length the A409

60. Ellis, "History," 318.

pitch refers. The nine instruments with c″ lengths of 357 to 371 mm could be characterized as 363 ± ⅓ semitone, which might be seen as one pitch level. In addition, since the three-instrument average of 345 is 88 cents shorter than the 363 average, the lower average could represent a pitch approximately one semitone higher. The 357-mm lengths could represent their own pitch level of A–1½, a level commonly found in eighteenth-century France, especially since the equation given above correlates the pitch level of A403 with a c″ length of 355 mm. Therefore, the possibilities include: (1) the tuning fork was meant to coordinate with the shorter scaling and happened to be pitched a little lower than A415; (2) the tuning fork was meant to coordinate with the longer scaling and happened to be pitched a little higher than A392; (3) one fork was meant to accommodate both the 345 and 363 groups, resulting in the shorter-scaled instruments being pitched a little lower than theoretical and the longer-scaled group pitched a little higher; or (4) the fork was specifically designed for A–1½ (average A403). These are too many possibilities to be useful as evidence.

Summary

Pitch levels were correlated with historical c″ string lengths in iron-strung harpsichords by establishing those levels independently of any consideration of tensile strength or safety factors. An equation was formulated that allows the calculation of an average A-based pitch level from c″ string lengths. The results indicate that a pitch of A415 correlates with an iron-strung c″ length of 345 mm, with a variance of ± ½ semitone. The benefit of this approach is that the information may help to identify or corroborate pitch levels for historical regions, makers, or instruments that are otherwise not clearly defined. The limitations of this approach include the following: (1) the results are based on average values and an average range of ± ½ semitone; (2) the makers' practices are unknown in terms of an acceptable range of design lengths for a given pitch or an acceptable pitch range for a given design length; and (3) an instrument may be tuned higher or lower than that dictated by an average pitch-length correlation.

APPENDIX:

Equations

Stress. Stress is calculated from fL values using the following equation:

$$\sigma = 4\rho(\mathrm{fL})^2$$

in which σ is tensile stress in N/m^2 (divide by 1,000,000 to get MPa), f is frequency in Hz, L is string length in meters, and ρ is density of the wire in kg/m^3 (for iron this value is 7800, for yellow brass 8500, for red brass 8700).

Semitones and cents. The number of equal-tempered semitones between two values is calculated using the following equation:

$$\text{Semitones} = 39.86 \times \log\,(x \div y)$$

in which x and y are any two quantities such as frequencies, lengths, or fL values. The logarithm is base 10. If x is greater than y, the result is a positive number; if x is less than y, the result is a negative number. For example, if x = 493 Hz and y = 415 Hz, the semitone value is +3, meaning that 493 is three semitones above 415. If x = 323 mm and y = 363 mm, the semitone value is –2, meaning a string length of 323 mm is two semitones shorter than a string length of 363 mm. To obtain cents, multiply the number of semitones by 100, or use 3986 instead of 39.86. It should be noted that the use of this semitone equation for relating values that are derived from the *squaring* of frequency, length, or fL—such as stress—results in semitone values that are *double* the unsquared values.

An alternate form of the semitone equation is:

$$\text{Multiplier} = 10^{(\text{semitones} \times 0.02509)}$$

which gives the multiplier used to find a certain number of semitones above or below a given value. For example, the +3 semitone multiplier is 1.189, so a frequency three semitones above 415 Hz is 415 × 1.189 = 493 Hz. Likewise, the –2 semitone multiplier is 0.8909, so a string length two semitones shorter than 363 mm is 363 × 0.8909 = 323 mm.